Lecture by Dr. Enzo Bonacci on the multidimensional extension of the Cosine Law in Mathematics

Enzo Bonacci was born in Brescia (Italy) in 1972 and spent there his childhood. At the end of the 70's his family moved to Latina, city where he still lives and works; his school marks were so excellent to deserve the City Medal conferred by the Mayor. During his scientific high school he received a prize that used to study in Cambridge (UK), where he was extremely impressed with Newton's manuscripts on maths and physics. After graduating in Chemical Engineering from "La Sapienza" University of Rome, he spent his university prize to travel the world and to achieve diplomas in numerous foreign languages. He was chosen to do his national service at the office of the Under Secretary of Defence. In spite of his scientific education he has never neglected his artistic side, writing poems and novels selected by international literary contests and becoming a columnist for some newspapers. Member of the *ODI* (Italian Order of Engineers) since 2001, he has become technical-scientific consultant for important boards. After qualifying in *mathematics* and *physics*, he has been teaching at Scientific High School since 2001, holding several posts like *Responsible for Public Relations* and *Secretary of the School Council*. In November 2003 he became responsible for the scientific project *Evolution of Rational Thinking and Epistemological Problems*. During 2004 he became responsible for the IFTS project *Transformation of Agroindustrial Products*. In January 2005 he was elected *Secretary of AEDE-Latina* (European Association of Teachers). In October 2007 he got the cover of BLU magazine about his effort to extend Relativity and became member of the *IOP* (MInstP). In 2008 he was selected among the 280 CBEL mathematicians and he was awarded with the Honorary Ph.D. in Theoretical Physics by the Cosmopolitan University.

Multidimensional extension of the Cosine Law

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Abstract

Some binomial properties permit to extend the Cosine Law to powers greater than two, *i.e.*, on synclastic surfaces (cf. ref.[1]).

1 Binomial properties for even numbers $2n>2$

Let us recall the Pascal's triangle through the binomial coefficients $C_{i,j}$ until i=9:

$C_{0,0}=1$

$C_{1,0}=1 \quad C_{1,1}=1$

$C_{2,0}=1 \quad C_{2,1}=2 \quad C_{2,2}=1$

$C_{3,0}=1 \quad C_{3,1}=3 \quad C_{3,2}=3 \quad C_{3,3}=1$

$C_{4,0}=1 \quad \boxed{C_{4,1}=4 \quad C_{4,2}=6 \quad C_{4,3}=4} \quad C_{4,4}=1$

$C_{5,0}=1 \quad C_{5,1}=5 \quad C_{5,2}=10 \quad C_{5,3}=10 \quad C_{5,4}=5 \quad C_{5,5}=1$

$C_{6,0}=1 \quad \boxed{C_{6,1}=6 \quad C_{6,2}=15 \quad C_{6,3}=20 \quad C_{6,4}=15 \quad C_{6,5}=6} \quad C_{6,6}=1$

$C_{7,0}=1 \quad C_{7,1}=7 \quad C_{7,2}=21 \quad C_{7,3}=35 \quad C_{7,4}=35 \quad C_{7,5}=21 \quad C_{7,6}=7 \quad C_{7,7}=1$

$C_{8,0}=1 \quad \boxed{C_{8,1}=8 \quad C_{8,2}=28 \quad C_{8,3}=56 \quad C_{8,4}=70 \quad C_{8,5}=56 \quad C_{8,6}=28 \quad C_{8,7}=8} \quad C_{8,8}=1$

$C_{9,0}=1 \quad C_{9,1}=9 \quad C_{9,2}=36 \quad C_{9,3}=84 \quad C_{9,4}=126 \quad C_{9,5}=126 \quad C_{9,6}=84 \quad C_{9,7}=36 \quad C_{9,8}=9 \quad C_{9,9}=1$

…

For binomial coefficients $C_{i,j}$ with $i=2n>2$ we may notice that:
if $2n=4$ then $C_{4,1}+C_{4,2}+C_{4,3}=14<4\cdot 2^2=16$;
if $2n=6$ then $C_{6,1}+C_{6,2}+C_{6,3}+C_{6,4}+C_{6,5}=62<6\cdot 2^4=96$;
if $2n=8$ then $C_{8,1}+C_{8,2}+C_{8,3}+C_{8,4}+C_{8,5}+C_{8,6}+C_{8,7}=254<8\cdot 2^6=512$;
and so on…
Let us sum up the above formulae as follows:
$0<C_{2n,1}+C_{2n,2}+ \cdots +C_{2n,n}+ \cdots +C_{2n,2n-2}+C_{2n,2n-1}<2n\cdot 2^{2n-2}$;
thus we get the first relevant binomial property:

$$\forall n,k\in \mathbf{N},\ n>1,\ 0<k<2n:\ 0<\sum_k C_{2n,k}/(2n\cdot 2^{2n-2})<1 \qquad (1.1)$$

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For binomial coefficients $C_{i,j}$ with i=2n>2 we may also notice that:
if 2n=4 then $-C_{4,1}+C_{4,2}-C_{4,3}=-2$;
if 2n=6 then $-C_{6,1}+C_{6,2}-C_{6,3}+C_{6,4}-C_{6,5}=-2$;
if 2n=8 then $-C_{8,1}+C_{8,2}-C_{8,3}+C_{8,4}-C_{8,5}+C_{8,6}-C_{8,7}=-2$;
and so on…
Let us sum up the above formulae as follows:
$-C_{2n,1}+C_{2n,2}-\cdots\pm C_{2n,n}\pm\cdots+C_{2n,2n-2}-C_{2n,2n-1}=-2$;
thus we get the second relevant binomial property:
$\boldsymbol{\forall n,k\in \mathrm{N},\ n>1,\ 0<k<2n:\ -1<\sum_k(-1)^k\cdot C_{2n,k}/(2n\cdot 2^{2n-2})<0}$ **(1.2)**

2 The Cosine Law extended to superior powers

Denote a,b,c the three sides of a flat triangle, being $a,b,c>2$.
By triangular properties, we have:
$c-b<a<b+c$;
$(c-b)^{2n}<a^{2n}<(b+c)^{2n}$.
By expanding the binomials, $\forall n,k\in \mathrm{N}$, $0<k<2n$:
$c^{2n}+b^{2n}-\sum_k(-1)^k\cdot C_{2n,k}\cdot b^{2n-k}\cdot c^k<a^{2n}<c^{2n}+b^{2n}-\sum_k C_{2n,k}\cdot b^{2n-k}\cdot c^k$.
According to the binomial properties 1.1 and 1.2, we have:
$0<\sum_k C_{2n,k}\cdot b^{2n-k}\cdot c^k/[2n\cdot(bc)^{2n-1}]<\sum_k C_{2n,k}/(2n\cdot 2^{2n-2})<1$;
$-1<\sum_k(-1)^k\cdot C_{2n,k}/[2n\cdot(2^{2n-2})]<\sum_k(-1)^k\cdot C_{2n,k}\cdot b^{2n-k}\cdot c^k/[2n\cdot(bc)^{2n-1}]<0$.
Hence there exist two values, whose moduli are inferior to the unity, interpretable as cosines of two angles $\alpha_1\in(0,\pi/2)$ and $\alpha_0\in(\pi/2,\pi)$:
$\cos\alpha_1=\sum_k C_{2n,k}\cdot b^{2n-k}\cdot c^k/[2n\cdot(bc)^{2n-1}]$;
$\cos\alpha_0=\sum_k(-1)^k\cdot C_{2n,k}\cdot b^{2n-k}\cdot c^k/[2n\cdot(bc)^{2n-1}]$;
therefore:
$b^{2n}+c^{2n}-2n\cdot(bc)^{2n-1}\cdot\cos\alpha_0<a^{2n}<b^{2n}+c^{2n}-2n\cdot(bc)^{2n-1}\cdot\cos\alpha_1$.
By comparing, there is an angle $\alpha_{2n}\in(\alpha_0,\alpha_1)$ such that:
$a^{2n}=b^{2n}+c^{2n}-2n\cdot(bc)^{2n-1}\cdot\cos\alpha_{2n}$.
Since a is any side, the Cosine Law is extendable to any 2n index:
$\boldsymbol{\forall n\in \mathrm{N},\ a,b,c>2,\ \exists\alpha_{2n},\beta_{2n},\gamma_{2n}\in(0,\pi)}$**:**
$\boldsymbol{a^{2n}=b^{2n}+c^{2n}-2n\cdot(bc)^{2n-1}\cdot\cos\alpha_{2n}}$ **(2.1)**
$\boldsymbol{b^{2n}=a^{2n}+c^{2n}-2n\cdot(ca)^{2n-1}\cdot\cos\beta_{2n}}$ **(2.2)**
$\boldsymbol{c^{2n}=a^{2n}+b^{2n}-2n\cdot(ab)^{2n-1}\cdot\cos\gamma_{2n}}$ **(2.3)**

3 The Cosine Law extended on synclastic surfaces

Let $K(\Delta)$ represent the *Gaussian curvature* of a geodesic triangle: $K(\Delta)=\alpha+\beta+\gamma-\pi$ (α,β,γ interior angles of Δ). *Synclastic* is a surface on which the Gaussian curvature is everywhere *positive*: $K>0$.
The triple of angles deriving from the 2*n-dimensional* extension of the Cosine Law (2.1-2.3):

$$\boldsymbol{\alpha_{2n}=\arccos\{(b^{2n}+c^{2n}-a^{2n})/[2n\cdot(cb)^{2n-1}]\}} \quad \textbf{(3.1)}$$

$$\boldsymbol{\beta_{2n}=\arccos\{(a^{2n}+c^{2n}-b^{2n})/[2n\cdot(ca)^{2n-1}]\}} \quad \textbf{(3.2)}$$

$$\boldsymbol{\gamma_{2n}=\arccos\{(a^{2n}+b^{2n}-c^{2n})/[2n\cdot(ab)^{2n-1}]\}} \quad \textbf{(3.3)}$$

belongs to triangles Δ_{2n} with same sides $a,b,c>2$ but variable curvatures $K(\Delta_{2n})\geq 0$.
If $n=1$ the triangle Δ_2 is flat, *i.e.*, at null curvature $K(\Delta_2)=0$.
If $n>1$ the triangle Δ_{2n} is geodetic, *i.e.*, at positive curvature $K(\Delta_{2n})>0$.
If $n\to\infty$ the triangle Δ_∞ is degenerate, *i.e.*, at curvature $K(\Delta_\infty)=\pi/2$, with $\alpha_\infty=\beta_\infty=\gamma_\infty=90°$.
Therefore while n grows from 1 to ∞ there is the passage from a flat surface, *i.e.*, in a null curvature space $K(\Delta_2)=0$, to a synclastic surface, *i.e.*, in a positive curvature space $K(\Delta_{2n})>0$.

4 Multidimensional increase from $\alpha_2, \beta_2, \gamma_2<90°$

In Figure 4.1 we have the basic triangle, with sides $a,b,c>2$ and angles $\alpha_2,\beta_2,\gamma_2<90°$.

Fig. 4.1

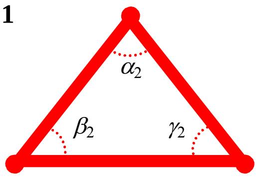
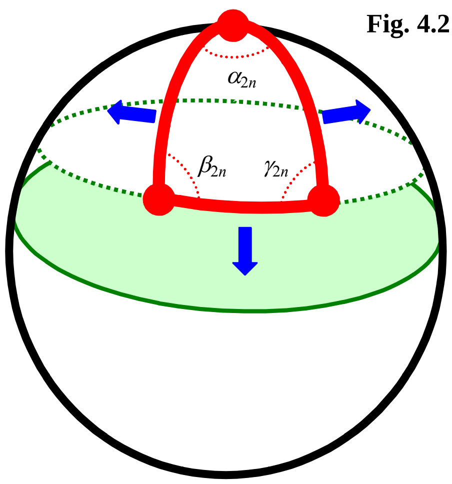


Fig. 4.2

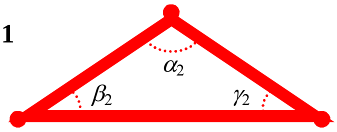


By 3.1-3.3, in Fig. 4.2
$\forall n\in \mathrm{N}$, $n>1$:
$\alpha_2<\alpha_{2n-1}<\alpha_{2n}<90°$
$\beta_2<\beta_{2n-1}<\beta_{2n}<90°$
$\gamma_2<\gamma_{2n-1}<\gamma_{2n}<90°$

5 Multidimensional increase from $\alpha_2 \geq 90°$

In Figure 5.1 we have the basic triangle, with sides $a,b,c>2$ and an angle $\alpha_2 \geq 90°$.

Fig. 5.1



By 3.1-3.3, in Fig. 5.2
$\forall n \in \mathrm{N}$, $n>1$:
$90° \leq \alpha_{2n} \leq \alpha_{2n-1} \leq \alpha_2$
$\beta_2 < \beta_{2n-1} < \beta_{2n} < 90°$
$\gamma_2 < \gamma_{2n-1} < \gamma_{2n} < 90°$

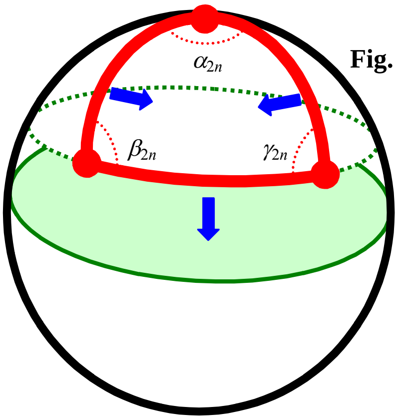


Fig. 5.2

6 Degenerate triangle Δ_∞

With growing n any flat triangle Δ_2 with sides $a,b,c>2$ (Fig. 6.1 and 6.2) degenerates in a triangle Δ_∞ (Fig. 6.3) whose inner angles are $\alpha_\infty=\beta_\infty=\gamma_\infty=90°$ and whose curvature is $K(\Delta_\infty)=\alpha+\beta+\gamma-\pi=\pi/2$.

Fig. 6.1

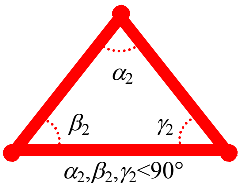
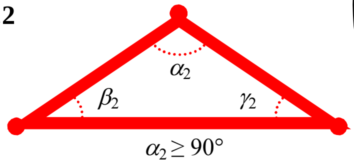


Fig. 6.2



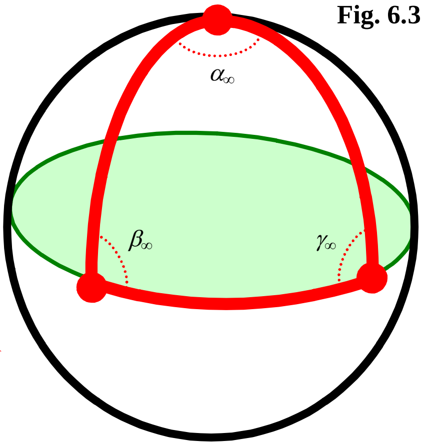


Fig. 6.3

7 Conclusions

An ordinary triangle lies on a flat surface (Fig. 7.1), *i.e.*, has a 2-dimensional curvature $K(\Delta_2)=\alpha+\beta+\gamma-\pi=0$, and its Cosine Law is bidimensional as well:

$a^2=b^2+c^2-2cb\cdot\cos\alpha_2$
$b^2=a^2+c^2-2ca\cdot\cos\beta_2$
$c^2=a^2+b^2-2ab\cdot\cos\gamma_2$

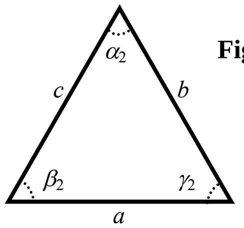


Fig. 7.1

A $2n$-dimensional triangle with sides $a,b,c>2$ and $n>1$ it lies on a synclastic surface (Fig. 7.2), *i.e.*, has a $2n$-dimensional curvature $K(\Delta_2)=\alpha+\beta+\gamma-\pi>0$, and its extended Cosine Law (2.1-2.3) is $2n$-dimensional as well:

$a^{2n}=b^{2n}+c^{2n}-2n[(cb)^{2n-1}]\cdot\cos\alpha_{2n}$
$b^{2n}=a^{2n}+c^{2n}-2n[(ca)^{2n-1}]\cdot\cos\beta_{2n}$
$c^{2n}=a^{2n}+b^{2n}-2n[(ab)^{2n-1}]\cdot\cos\gamma_{2n}$

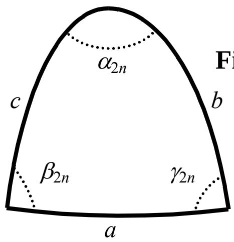


Fig. 7.2

A ∞-dimensional triangle with sides $a,b,c>2$ lies on a synclastic surface (Fig. 7.3), *i.e.*, has the ∞-dimensional curvature $K(\Delta_\infty)=\alpha+\beta+\gamma-\pi=\pi/2$, and its degenerate Cosine Law (2.1-2.3) is ∞-dimensional as well:

$a^\infty=b^\infty+c^\infty$
$b^\infty=a^\infty+c^\infty$
$c^\infty=a^\infty+b^\infty$

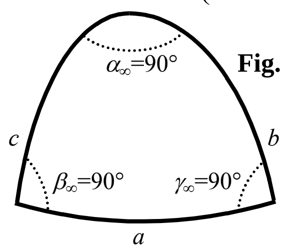


Fig. 7.3

Acknowledgements

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References

[1] Bonacci E., "Multidimensional Extension of the Cosine Law," *Periodico di Matematiche* 1 (2008), Pages: 107-111